Symmetry-increasing bifurcation as a predictor of a chaos-hyperchaos transition in coupled systems

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In weakly coupled systems, it is possible to observe the coexistence of the chaotic attractors which are located out of the invariant manifold and are not symmetrical in relation to this manifold. When the control parameter is changed, these attractors can undergo a chaos-hyperchaos transition. We give numerical evidence that before this transition the coexisting attractors merge together creating an attractor symmetrical with respect to the invariant manifold. We argue that the attractors that are not located at the invariant manifold can exhibit dynamical behavior similar to bubbling and on-off intermittency previously observed for the attractors located at the invariant manifold, and we describe the mechanism of these phenomena.

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Nowadays we can observe a growing interest in higher-dimensional dynamical systems. Consider a dynamical system given by a flow $\dot{x}=f(a,x)$, where $x\in\mathcal{R}^m$ and $a\in\mathcal{R}$ is a control parameter. Such a system is characterized by m Lyapunov exponents $\lambda_1,\dots,\lambda_m$, where $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_m$. If $\Sigma_{i=1}^m\lambda_i<0$, the system is dissipative and its evolution takes place on an attractor. The attractors that are characterized by only one positive Lyapunov exponents are called chaotic and they can occur in at least three-dimensional systems. Generally, for m-dimensional dissipative systems one can observe attractors with $m-2$ positive Lyapunov exponents. The attractors that are characterized by at least two positive Lyapunov exponents for typical trajectories on them are called hyperchaotic [1].

The first example of such an attractor was presented by Rössler [2] for a model of the particular chemical reaction. Later, hyperchaotic attractors were found in electronic circuits and other chemical reactions [3]. In [4], it was shown that by weak coupling of m chaotic systems, it is possible to obtain a hyperchaotic attractor with m positive Lyapunov exponents. The transition from chaos to hyperchaos, i.e., the bifurcation that occurs when the second largest Lyapunov exponent λ_2 becomes positive, has been studied in [5]. It has been shown that at this transition the attractor dimension and λ_2 grow continuously.

On the other hand, for Hamiltonian systems we have a condition $\Sigma_{i=1}^m\lambda_i=0$, and although the evolution of such systems is not restricted to attractors, it is possible to observe trajectories characterized by more than one positive Lyapunov exponents. For example, a three-dimensional Hamiltonian system can have one or two positive exponents [the cases with only one positive Lyapunov exponent are those with two (not three) integrals of motion] [6]. A distinction between ''strong chaos'' (hyperchaos according to our terminology) with $\lambda_{1,2}>0$ and ''weak chaos'' (chaos) with $\lambda_1>0$ and $\lambda_2=0$ was made by Pettini and Vulpiani [7].

In a previous paper [8], we studied the dynamical system given by a dissipative map $x_{n+1}=f(a,x_n)$, where $x\in\mathcal{R}^2$ and $a\in\mathcal{R}$. In such systems, due to the stretching and folding mechanism, one can observe attractors with one or two positive Lyapunov exponents. Generally if such a map is m-dimensional ($x\in\mathcal{R}^m$), one can observe attractors with m positive Lyapunov exponents. We assumed that the system evolved on the chaotic attractor A (i.e., with one positive Lyapunov exponent) and allowed the control parameter to vary slowly in such a way that the second Lyapunov exponent became positive and thus the attractor A became hyperchaotic. We gave evidence that the chaos-hyperchaos transition was typically stretching (spreading) along the control parameter interval, and that its mechanism had the same characteristic features as the blowout bifurcation of the attractors located at invariant manifolds in systems with symmetry [9].

In this paper, we consider weakly coupled continuous systems in which it is possible to observe the coexistence of the chaotic attractors (say $\mathcal{A}$ and $\mathcal{B}$) which are located out of the invariant manifold and are not symmetrical in relation to this manifold. When the control parameter is changed, these attractors can undergo a chaos-hyperchaos transition (at $a=a_h$). We give numerical evidence that before this transition the coexisting attractors $\mathcal{A}$ and $\mathcal{B}$ merge together (at $a=a_s<a_h$) creating the attractor $\mathcal{C}$, which is symmetrical with respect to the invariant manifold.

As an example, consider two identical symmetrically coupled Rössler systems,

$$\begin{aligned}
\dot{x}_1&=-x_2-x_3,\\
\dot{x}_2&=x_1+ax_2,\\
\dot{x}_3&=b+x_3(x_1-c)+d(y_3-x_3),\\
\dot{y}_1&=-y_2-y_3,\\
\dot{y}_2&=y_1+ay_2,\\
\dot{y}_3&=b+y_3(y_1-c)+d(y_3-x_3),
\end{aligned}\tag{1}$$

where $(x_1,x_2,x_3,y_1,y_2,y_3)\in\mathcal{R}^6$ are dynamical variables; a,b,c are constant system parameters, and d is the coeffi-

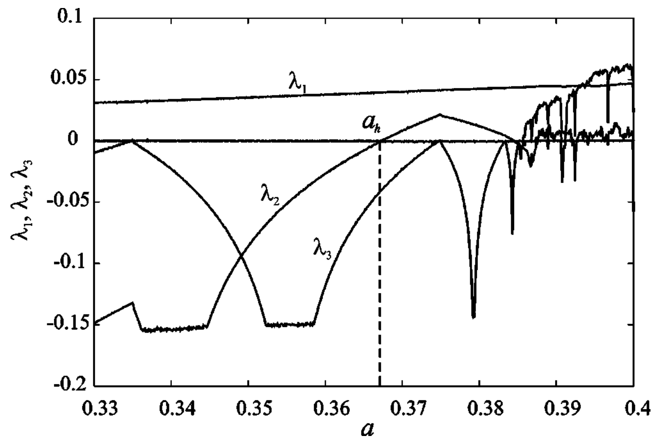

FIG. 1. Variation of three largest Lyapunov exponents for the coupled Rössler system for $d=0.25$. The smooth transition to hyperchaos occurs at $a_h \approx 0.3673$.

cient of coupling. It is well known that the Rössler system develops continuous chaos through a period-doubling bifurcation cascade [2]. Since the Rössler system has a foundation in the kinematics of the chemical reaction [2], it is natural to study the diffusive coupling of two such systems [10].

In our numerical studies we took the parameter values $b=2.0$, $c=4.0$, and $d=0.25$, and we considered a as a control parameter. With an increase in the control parameter a, system (1) reveals the transition to hyperchaos [10] with a smooth passage of the second Lyapunov exponent through zero. A variation of the three largest nonzero Lyapunov exponents versus a is shown in Fig. 1 [for system (1), one of the Lyapunov exponents has to be equal to zero]. One can observe a typical smooth transition to hyperchaos (similar to that observed in [5,8]) at $a_h \approx 0.3673$. It should be noted here that the chaos-hyperchaos transition at a_h is not the only transition of this type. For larger a, the second Lyapunov exponent decreases and then increases again through zero. These further hyperchaos-chaos and chaos-hyperchaos transitions are not studied here in particular but their mechanism is the same as the one described in this paper.

The chaos-hyperchaos transition is mediated by an infinite number of unstable periodic orbits. In the neighborhood of the transition point, we observe the coexistence of two classes of UPO's. UPO's of the first class have exactly one unstable eigenvalues while UPO's of the second class have at least two unstable eigenvalues [14]. [As was shown in [1], the six-dimensional system (1) can have up to four unstable eigenvalues.] This coexistence is responsible for the occurrence of nonhyperbolic behavior known as unstable dimension variability and can explain the smooth passage through zero of the second Lyapunov exponent at the chaos-hyperchaos transition point [11,8,14].

Before the chaos-hyperchaos transition point a_h, system (1) has two coexisting chaotic attractors located out of the invariant manifold. The corresponding Poincaré section illustrates both the attractors in Fig. 2. The attractor $\mathcal{A}$ is shown

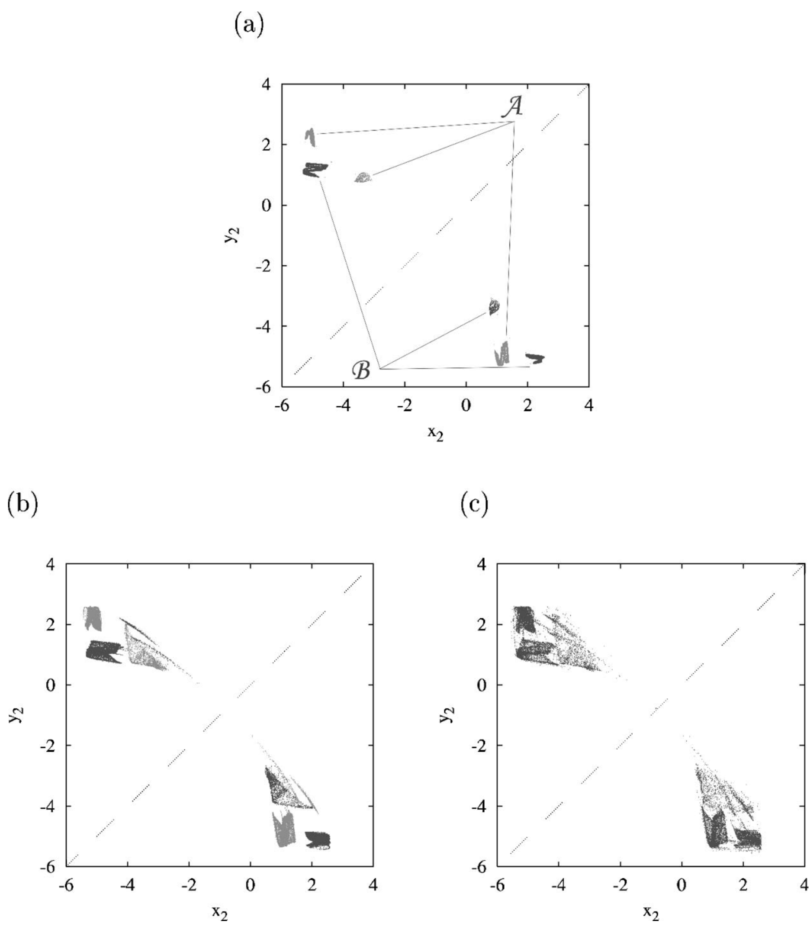

FIG. 2. Symmetric Poincaré cross section illustrates how two symmetric attractors are merged together at $a \approx 0.367\,15$.

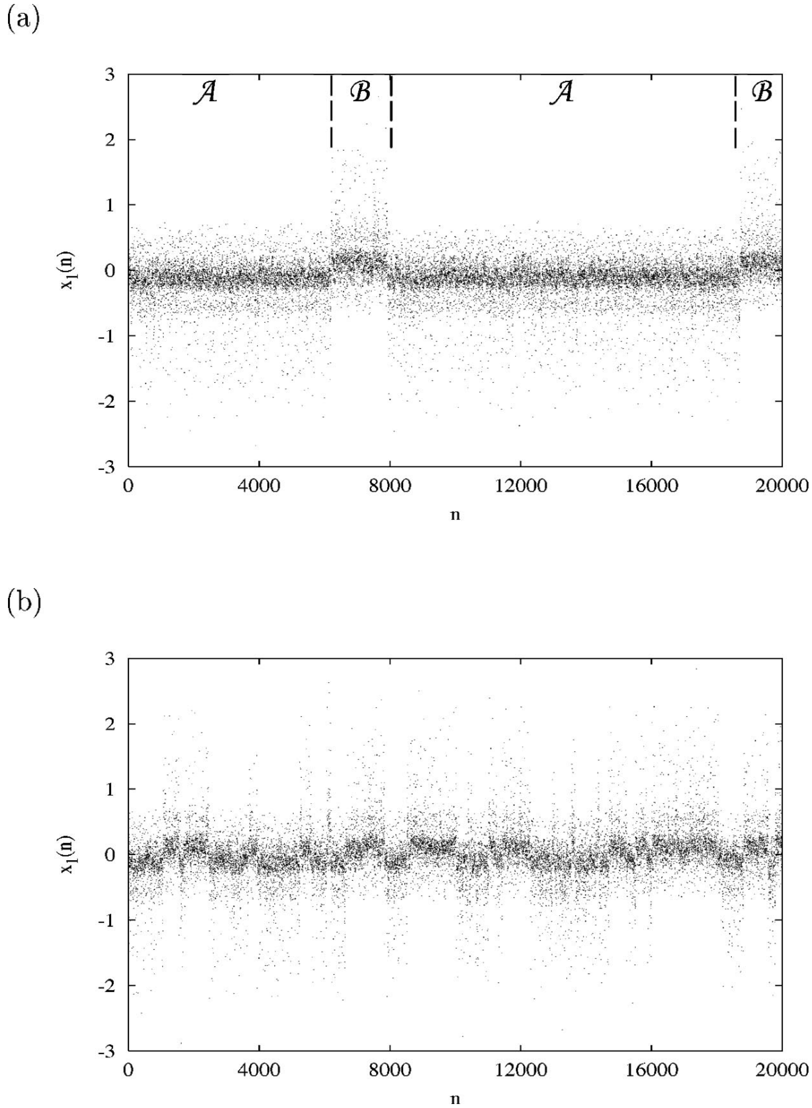

FIG. 3. Switching behavior of x_1 component of the Poincaré map for $a=0.3672$ (a) and $a=0.3674$ (b) ($n=t/h$, where h is an integration step).

in gray, whereas the attractor $\mathcal{B}$ is shown in black. Both the attractors $\mathcal{A}$ and $\mathcal{B}$ are not symmetrical in relation to the invariant manifold $x=y$. Note that the cross section has been chosen in such a way that it contains the origin and is normal to the symmetric vector $(1,0,0,1,0,0)$. Thus, the sets that are symmetrical with respect to the invariant manifold $x=y$ will have also the symmetric images on the Poincaré map with respect to the diagonal $x_2=y_2$ in Fig. 2. The calculations reveal two distinct attractors at $a=0.363$ [Fig. 2(a)] and $a=0.367$ [Fig. 2(b)], whereas for $a>a_s\approx 0.367\,15$ there exists only one invariant set $\mathcal{C}$ which unites both the previously distinct attractors $\mathcal{A}$ and $\mathcal{B}$ shown in Fig. 1(c) for $a=0.368$. As the attractor $\mathcal{C}$ created after the merging of the attractors $\mathcal{A}$ and $\mathcal{B}$ is symmetrical with respect to the $x=y$ manifold, it is justified to call the bifurcation that occurs at a_s a *symmetry-increasing bifurcation*. The attractor $\mathcal{C}$ is still chaotic as $a_s<a_h$.

It has been observed that shortly after the symmetry-increasing bifurcation, system (1) exhibits intermittencylike behavior on the attractor $\mathcal{C}$ with a relatively low number of switches between the previously coexisting attractors $\mathcal{A}$ and $\mathcal{B}$. Such behavior can be observed in Figs. 3(a) and 3(b) for the parameter values of a close to a_h ($a=0.3672<a_h$, Fig. 3(a) and $a=0.3674>a_h$, Fig. 3(b)]). In Fig. 3(a), the system trajectory x_1 spends most of the time in the phase-space region, where the attractor $\mathcal{A}$ was located, and only occasionally jumps to the region of the attractor $\mathcal{B}$, whereas in Fig. 3(b) it jumps between the former attractors $\mathcal{A}$ and $\mathcal{B}$ more frequently.

The mechanism of this phenomenon can be explained using an analogy to the attractor bubbling and on-off intermittency for the case of a chaotic set belonging to the invariant manifold $x=y$ [12]. Following [8], we shall distinguish two different control parameter intervals—$a<a_r$ and $a_r<a<a_h$—before the chaos-hyperchaos transition. In the first interval, all unstable periodic orbits embedded into the chaotic attractor have only one unstable dimension (the second largest modulus of multipliers is smaller than 1). At the end of this interval at $a=a_r$ the first unstable periodic orbit embedded into the chaotic attractor becomes doubly unstable, i.e., the modulus of the second largest multiplier increases through 1. The second interval $a_r<a<a_h$ corresponds to the case in which the attractor still has only one positive Lyapunov exponent with respect to the natural mea-

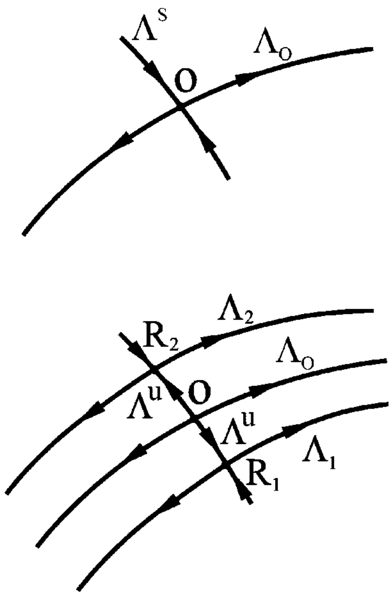

FIG. 4. Scheme of possible bifurcations when the fixed point of the Poincaré map becomes doubly unstable at a_r.

sure but there are unstable orbits with a different number of unstable directions embedded into it (unstable periodic orbits with one and two unstable directions have been observed). The nonhyperbolic state in which the unstable periodic orbits with a different number of unstable directions coexist is known as unstable dimension variability [11]. Finally, the chaos-hyperchaos transition occurs at some parameter value a_h when the appropriate weights of unstable periodic orbits with one and at least two unstable directions are balanced, as was shown in [8].

Let us consider in detail the appearance of the first doubly unstable periodic orbit in the attractor, which occurs when the parameter a is increasing past a_r. In terms of the Poincaré map without loss of generality, we may consider a fixed point of the corresponding map embedded into a chaotic attractor and having the second multiplier crossing through unity at a_r in the modulus. A similar case was considered in [8]. While before the bifurcation both the fixed point and its unstable manifold belong to the attractor, cf. Fig. 4(a), after the bifurcation it is possible to observe how the unstable manifold Λ^u, which is directed transversely to the attractor, appears; cf. Fig. 4(b). Figure 4(b) illustrates only two possible cases when the fixed point O undergoes either period-doubling or saddle-node bifurcation. The first case leads to the appearance of period-2 cycle R_1, R_2, whereas the latter one leads to the appearance of two fixed points R_1 and R_2 and is generally realized when a system has additional symmetry properties. Note that the case considered in Fig. 4 generally leads to a soft increase in the attractor size. This follows from the fact that after the bifurcations the orbits starting in the vicinity of the doubly unstable point O will be bounded by the local saddle manifolds Λ_1 and Λ_2 of the created periodic orbit R_1, R_2 or fixed points R_1 and R_2.

Another possible scenario which has been observed in the considered system (1) is an absorption of a doubly unstable periodic cycle [8]. In this case an unstable manifold of the absorbed cycle and orbits nearby will diverge out of the attractor.

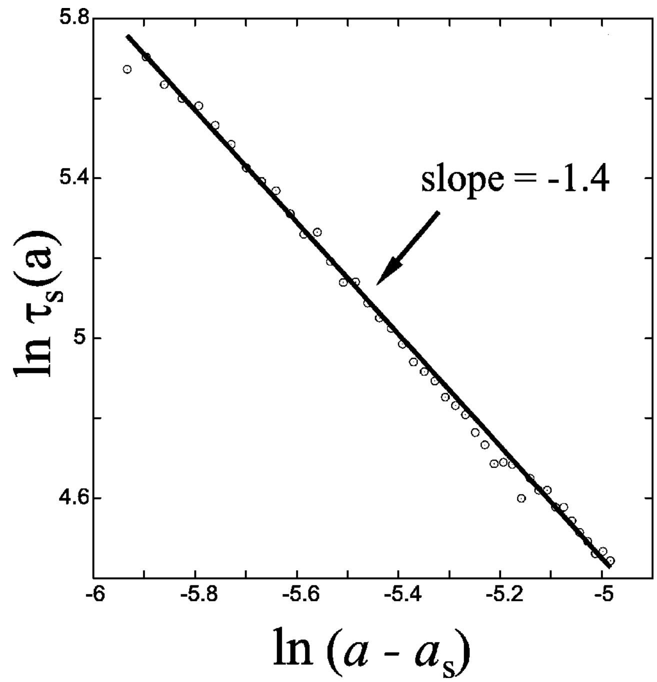

FIG. 5. For the system (1) (after the symmetry-increasing bifurcation), the average intermittent switching time $\tau_s(a)$ versus $(a - a_s)$. We can see that $\tau_s(a) \sim (a-a_s)^{-1.4}$.

The described scenarios create orbits that diverge from the attractor (say $\mathcal{A}$). The collision of such a divergent orbit with the boundary of the basin of attraction $\beta(\mathcal{A})$ causes the switching to the other attractor $\mathcal{B}$. In our example (1), due to the coexistence of the attractors $\mathcal{A}$ and $\mathcal{B}$ in the phase space, we observe intermittent behavior between both the attractors. In general, if there are no other attractors in the neighborhood of the attractor $\mathcal{A}$, then the initially repelled orbit will return to $\mathcal{A}$. Such behavior on the attractor located at the invariant manifold is known as bubbling (if it occurs before the blowout bifurcation) or on-off intermittency (if it occurs after the blowout bifurcation).

For the observed intermittency one can get the following scaling law. Let $\tau_s(a)$ be the average time that the system trajectory stays in the region of the phase space, where one of the coexisting attractors ($\mathcal{A}$ or $\mathcal{B}$) was located before the symmetry-increasing bifurcation. We investigate the scaling relation that is to occur between $\tau_s(a)$ and a control parameter a. Following [13], we expect to obtain the algebraic scaling relation

$$\tau_s(a) \sim (a-a_s)^{-\gamma}, \tag{2}$$

where a_s is a point of the symmetry-increasing bifurcation. Figure 5 shows $\tau_s(a)$ versus $(a-a_s)$ on a logarithmic scale for $a_s < a < 0.374$, where the number of switching events is calculated for each a for an orbit of length 3×10^5. We have approximately found the moment where the switching is initiated: $a_s \approx 0.367\,15 < a_h$. The data can be fitted by a straight line with a slope $\gamma \approx -1.4$.

To summarize, we have shown here that in weakly coupled systems the transition from chaos to hyperchaos occurs after the symmetry-increasing bifurcation. At this bifurcation, the coexisting chaotic attractors located out of the invariant manifold and nonsymmetrical in relation to this manifold merge together, creating a chaotic attractor that is symmetrical in relation to the invariant manifold. Both the symmetry-increasing bifurcation and the chaos-hyperchaos transition are caused by bifurcations of an infinite number of unstable periodic orbits. We have shown that in coupled systems, the attractors that are not located at the invariant manifold can exhibit dynamical behavior similar to bubbling and on-off intermittency previously observed for the attractors located at the invariant manifold. Finally, we believe that the observed phenomena are characteristic of coupled continuous systems (at least those in which chaos, as in the Rössler system, is observed after the cascade of period-doubling bifurcations).

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